Mathematics 181 – **D** **Fall 2006**

Exam 1

September 19, 2006

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Name

Technology used:_______________________ Directions:

- Be sure to include in-line citations every time you use technology.
- Include a careful sketch of any graph obtained by technology in solving a problem.
- **Only write on one side of each page.**

Do any six (6) of the following problems

1. (15 points) Use the **definition** of definite integrals as the limit of Riemann sums and the Useful Facts below to compute

$$\int_0^2 \left(12x^2 + 2x\right)\ dx.$$

[No credit for using the Fundamental Theorem of Calculus]

(a) Partition the interval $[0, 2]$ into n subintervals each of size $\Delta x = \frac{2-0}{n}$ so that $P = \left\{0, \frac{2}{n}, 2\frac{2}{n}, 3\frac{2}{n}, \cdots, n\frac{2}{n}\right\}$.

(b) Select the right endpoint $c_k = 2\frac{k}{n}$ in the k^{th} subinterval.

(c) Form a Riemann Sum by adding up the products $f(c_k)\Delta x$ obtaining

(d) $\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n}\left[12\left(\frac{2k}{n}\right)^2 + 2\left(\frac{2k}{n}\right)\right]\frac{2}{n}$.

(e) Simplify the Riemann Sum $\sum_{k=1}^{n}\left[12\left(\frac{2k}{n}\right)^2 + 2\left(\frac{2k}{n}\right)\right]\frac{2}{n}$

(f) $= \frac{2}{n}\left[12\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^2 + 2\sum_{k=1}^{n}\left(\frac{2k}{n}\right)\right] = \frac{96}{n^3}\sum_{k=1}^{n} k^2 + \frac{8}{n^2}\sum_{k=1}^{n} k$

(g) Use the appropriate formulas (discrete antiderivatives)

(h) $\sum_{k=1}^{n} f(c_k)\Delta x = \frac{96}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{8}{n^2}\cdot\frac{1}{2}n(n+1)$

(i) Simplify: $\sum_{k=1}^{n} f(c_k)\Delta x = 16\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 4\left(1+\frac{1}{n}\right)$

(j) Finish by finding $\int_0^2 \left(12x^2+2x\right)\ dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x$

(k) $= \lim_{n\to\infty} 16\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 4\left(1+\frac{1}{n}\right) = 16\cdot 1\cdot 2 + 4\cdot 1 = 32 + 4 = 36.$

(l) Check by using FTC: $\int_0^2 \left(12x^2+2x\right)\ dx = \left[4x^3 + x^2\right]_0^2 = \left[(32+4) - (0+0)\right] = 36$

2. (15 points) Do any **two** (2) of the following

(a) Use the definition (see Useful Facts below) to compute the discrete derivative of the following sequence $b(n) = (n+2)5^n$. (Use algebra to factor your answer.)

i. $D_n[b(n)] = b(n+1) - b_n = (n+3)5^{n+1} - (n+2)5^n$

ii. $= 5^n[5(n+3) - (n+2)] = (4n+13)5^n$

(b) Explain why

$$\sum_{k=1}^{n}\left(k^7+2k\right)=\sum_{j=4}^{n+3}\left((j-3)^7+2j-6\right)$$

i. Setting $k=j-3$ so that $j=k+3$ we see $\sum_{k=1}^{n}\left(k^7+2k\right)$
ii. $=\sum_{j-3=1}^{n+3}\left((j-3)^7+2(j-3)\right)=\sum_{j=4}^{n+3}\left((j-3)^7+2j-6\right)$.

(c) Express the following limit as a definite integral where P is a partition of the interval $[0,\frac{\pi}{3}]$

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n}\left(\tan(c_k)\ \Delta x_k\right)$$

i. $\int_0^{\frac{\pi}{3}}\tan(x)\ dx$

3. (8, 7 points) Evaluate the following indefinite integrals.

(a)

$$\int\left(2e^x+\frac{3}{x}+4\sec^2(x)-5\cos(x)\right)\ dx$$

i. $2\int e^x\ dx+3\int\frac{1}{x}\ dx+4\int\sec^2(x)\ dx-5\int\cos(x)\ dx$
ii. $=2e^x+3\ln|x|+4\tan(x)-5\sin(x)+C$

(b)

$$\int\frac{1}{u^4}\left(\frac{2}{u}-\frac{7}{u^3}+\sqrt[3]{u}\right)\ du$$

i. $\int\left(\frac{2}{u^5}-\frac{7}{u^7}+\frac{u^{1/3}}{u^4}\right)du=2\int u^{-5}\ du-7\int u^{-7}\ du+\int u^{-11/3}\ du$
ii. $=\frac{2}{-4}u^{-4}-\frac{7}{-6}u^{-6}+\frac{1}{-8/3}u^{-8/3}+C=-\frac{1}{2u^4}+\frac{7}{u^6}-\frac{3}{8u^{8/3}}+C$

4. (5 points each) Do all of the following

(a) What is the average value of the function $f(x)=x^5-7x^2+2$ on the interval $[2,6]$? [**Do not** use a Riemann Sum]

i. $\text{av}(f)=\frac{1}{6-2}\int_2^6\left(x^5-7x^2+2\right)\ dx=\frac{1}{4}\left[\frac{x^6}{6}-7\frac{x^3}{3}+2x\right]_2^6$
ii. $=\left[\left(\frac{6^6}{24}-7\frac{6^3}{12}+\frac{1}{4}2(6)\right)-\left(\frac{2^6}{24}-7\frac{2^3}{12}+\frac{1}{4}2(2)\right)\right]=1822$

(b) Given the function $f(x)=x^3+1$ with domain the interval $[0,4]$. Write a Riemann sum for f using a partition P that divides $[0,4]$ into 3 subintervals and where $\|P\|=2$. Be sure to specify P as well as writing out the three terms in the Riemann Sum.

i. There are infinitely many possible solutions. We use $P=\{0,2,3,4\}$ and right endpoints so that the first subinterval gives $\|P\|=2$. and $c_1=2$, $c_2=3$, $c_3=4$ and $\Delta x_1=2$, $\Delta x_2=1$, $\Delta x_3=1$
ii. Then $\sum_{k=1}^{3}f(c_k)\Delta x_k=(2^3+1)\,2+(3^3+1)\,1+(4^3+1)\,1$

(c) Suppose that f and g are integrable functions and that $\int_a^b(2f(x)+g(x))\ dx=5$ and $\int_a^b(f(x)-g(x))\ dx$ 7. Use properties of definite integrals to find $\int_a^b f(x)\ dx$ and $\int_a^b g(x)\ dx$. Show your work.

i. $\int f(x)\,dx-\int g(x)\,dx=7$ tells us $\int g(x)\,dx=\int f(x)\,dx-7$
ii. so $2\int_a^b f(x)\,dx+\int g(x)\,dx=5$ becomes $2\int f(x)\,dx+\int f(x)\,dx-7=5$

iii. which tells us $3\int f(x)\,dx = 12$ or $\int f(x)\,dx = 4$.

iv. Substitution now gives $\int g(x)\,dx = \int f(x)\,dx - 7 = 4 - 7 = -3$.

5. (8, 7 points) Do both of the following

(a) Find the derivative of

$$y = \int_{e^x}^{2} \tan^2(t)\; dt$$

i. $\int_{e^x}^{2} \tan^2(t)\; dt = -\int_{2}^{e^x} \tan^2(t)\; dt$ and using

ii. $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\,g'(x)$ we get $\frac{dy}{dx} = -e^x \tan^2(e^x)$

(b) Find the derivative of

$$y = \int_{x}^{x^2} \frac{1}{t}\, dt$$

i. Using Part 2 of the FTC: $\int \frac{1}{t}\,dt = \ln|t| + C$ so $y = \int_x^{x^2} \frac{1}{t}\,dt = \ln|t|]_x^{x^2}$

ii. $= \ln|x^2| - \ln|x| = \ln\left|\frac{x^2}{x}\right| = \ln|x|$. So $\frac{dy}{dx} = \frac{d}{dx}[\ln|x|] = \frac{1}{x}$.

6. (15 points) Use substitution to evaluate any **two** (2) of the following indefinite integrals

(a)

$$\int \frac{1}{\theta^2} \sin\left(\frac{1}{\theta}\right) \cos\left(\frac{1}{\theta}\right)\; d\theta$$

i. Setting $u = \frac{1}{\theta}$, then $du = \frac{-1}{\theta^2}d\theta$ and $\int \frac{1}{\theta^2}\sin\left(\frac{1}{\theta}\right)\cos\left(\frac{1}{\theta}\right)\; d\theta = -\int \sin(u)\cos(u)\,du$.

ii. Now setting $w = \sin(u)$, $dw = \cos(u)\,du$ we have

iii. $\int \frac{1}{\theta^2}\sin\left(\frac{1}{\theta}\right)\cos\left(\frac{1}{\theta}\right)\; d\theta = -\int \sin(u)\cos(u)\,du = -\int w\,dw =$

iv. $-\frac{1}{2}w^2 + C = -\frac{1}{2}\sin^2\left(\frac{1}{\theta}\right) + C$

(b)

$$\int \frac{\left(\sin^{-1} x\right)^2}{\sqrt{1-x^2}}\, dx$$

i. Setting $u = \arcsin(x)$, we obtain $du = \frac{1}{\sqrt{1-x^2}}dx$ so

ii. $\int \frac{\left(\sin^{-1} x\right)^2}{\sqrt{1-x^2}}\, dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\arcsin(x))^3 + C$

(c)

$$\int \frac{dx}{x\sqrt{x^4-1}}$$

i. Setting $u = x^2$ we have $du = 2xdx$ so we can deduce that $x = \sqrt{u}$ and $dx = \frac{1}{2x}du = \frac{1}{2\sqrt{u}}du$.

ii. Substituting: $\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{\frac{1}{2\sqrt{u}}du}{\sqrt{u}\sqrt{u^2-1}} =$

iii. $\frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{2}\operatorname{arcsec}(u) + C = \frac{1}{2}\operatorname{arcsec}(x^2) + C$

iv. Checking: $y = \frac{1}{2}\operatorname{arcsec}(x^2) + C$

v. has $\frac{dy}{dx} = \frac{1}{2}\frac{1}{x^2\sqrt{(x^2)^2-1}}\frac{d}{dx}[x^2] = \frac{1}{x\sqrt{x^4-1}}$

7. (15 points) The following is a list of the first few terms of a sequence $a(n)$ with domain $n = 0, 1, 2, \cdots$. Determine the formula for $a(n)$.

$$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, \cdots$$

[Hint: If $b(n)$ has terms $2, 5, 8, 11, 14, 17, 20, \cdots$, then the first few terms of the discrete derivative of $b(n)$ would be $(5-2), (8-5), (11-8), (14-11), (17-14), (20-17), \cdots$.But this is easily seen to be $3, 3, 3, 3, 3, 3, \cdots$. So Hence $D_n[b(n)] = c(n) = 3$.]

(a) The discrete derivative, $b(n)$, of $a(n)$ has terms

(b) $(1-2), (6-1), (17-6), (34-17), (57-34), (86-57), \cdots,$

(c) This simplifies to $-1, 5, 11, 17, 23, 29, \cdots$,

(d) The discrete derivative, $c(n)$, of $b(n)$ has terms

(e) $(5-1), (11-5), (17-11), (23-17), (29-23), \cdots$,

(f) So $c(n) = 6$ with domain $n = 0, 1, 2, 3, \cdots$

(g) The discrete antiderivatives of $c(n)$ look like $b(n) = 6n + C$

(h) but since $b(0) = -1 = 6(0) + C$ we see $C = -1$ and

(i) $b(n) = 6n - 1$, with domain $n = 0, 1, 2, 3, \cdots$

(j) The discrete antiderivatives of $b(n)$ look like $a(n) = 6 \cdot \frac{1}{2} n^{\underline{2}} - n + C$

(k) but since $a(0) = 2 = 6 \cdot \frac{1}{2} 0 (0-1) - 0 + C$ we see $C = 2$ and

(l) $a(n) = 6 \cdot \frac{1}{2} n^{\underline{2}} - n + 2$, with domain $n = 0, 1, 2, 3, \cdots$.

(m) Simplifying we get $a(n) = 6 \cdot \frac{1}{2} n^{\underline{2}} - n + 2 = 3n(n-1) - n + 2 = 3n^2 - 4n + 2$

Useful Facts

1. •

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

- $D_n[a(n)] = a(n+1) - a(n)$
- $n^{\underline{p}} = n(n-1)(n-2)\cdots(n-p+1)$
- $D_n[n^{\underline{p}}] = pn^{\underline{p-1}}$ and If $a(n) = n^{\underline{p}}$, then $A(n) = \frac{1}{p+1} n^{\underline{p+1}} + C$
- $D_n[r^n] = (r-1) r^n$ and if $a(n) = r^n$ then $A(n) = \frac{1}{r-1} r^n + C$
- $\sum_{k=m}^{n} a(k) = A(n+1) - A(m)$